Analysis on Manifolds (WBMA013-05)– Final Exam

Tuesday 26 January 2021, 8:30h–12:30h

This exam consists of **3** problems.
Usage of the theory and examples from the lecture notes is allowed with the only exception of the results of Exercise 4.1.13 from the lecture notes. Give a precise reference to the theory and/or exercises you use for solving the problems.
You get 10 points for free.

Problem 1. (9 + 15 + 6 = 30 points)

Let $f(x, y, z) = x^2 + y^2 + z^2$ and let $\sigma : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$\sigma(u, v) = \frac{1}{u^2 + v^2 + 1}\left(2u, 2v, 1 - u^2 - v^2\right). \tag{1}$$

(a) If $X = y\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}$, compute $\iota_X df$.

(b) Verify that $d(f \circ \sigma) = \sigma^* df$ by computing separately both terms.

(c) Does $du \wedge \sigma^* df$ define a volume form on $\mathbb{R}^2$? If so, is it positively oriented with respect to the standard euclidean basis? Justify your answer.

Problem 2. (6 + 6 + 6 + 6 + 6 = 30 points)

Recall that we can identify the space $\mathrm{Mat}(2, \mathbb{R})$ of 2×2-matrices with $\mathbb{R}^4$ by associating the matrix $X = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$ with the point $(x_{11}, x_{12}, x_{21}, x_{22}) \in \mathbb{R}^4$.

(a) Show that the set

$$\mathrm{SL}(2, \mathbb{R}) = \{A \in \mathrm{Mat}(2, \mathbb{R}) \mid \det A = 1\}$$

is a 3-dimensional smooth submanifold of $\mathrm{Mat}(2, \mathbb{R})$.
Hint: use the identification between matrices and $\mathbb{R}^4$.

(b) Let $e \in \mathrm{Mat}(2, \mathbb{R})$ denote the identity matrix. Show that

$$T_e\mathrm{SL}(2, \mathbb{R}) = \{A \in \mathrm{Mat}(2, \mathbb{R}) \mid \mathrm{tr}\, A = 0\},$$

where $\mathrm{tr} A$ denotes the matrix trace, i.e., the sum of the diagonal entries of A.

(c) Let $\iota : \mathrm{SL}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$ be the map $\iota(A) = A^{-1}$. Show that ι is smooth.

(d) Show that $d\iota_e : T_e\mathrm{SL}(2, \mathbb{R}) \to T_e\mathrm{SL}(2, \mathbb{R})$ is given by $d\iota_e(A) = -A$.

(e) Show that $SL(2, \mathbb{R})$ is a Lie group and give its Lie algebra.

Problem 3. (6 + 6 + 10 + 8 = 30 points)

In this problem we are going to prove a pair of important theorems about fixed points. We first prove the following.

Theorem 1. *Let P be a compact n-dimensional submanifold of $\mathbb{R}^n$ with non-empty boundary ∂P. Then, there is no differentiable map $\psi : P \to \partial P$ for which every boundary point is a fixed point, that is, for which $\psi(p) = p$ for all $p \in \partial P$.*

Let $\Theta = dx^1 \wedge \cdots \wedge dx^n$ denote the standard volume form on P, that is, the restriction of the standard volume form on $\mathbb{R}^n$ to P, and Z be an outward-pointing vector field on ∂P.

(a) Show that $\theta = \iota_Z \Theta$ is a closed non-vanishing form on ∂P.

(b) Show that if there exists $\psi : P \to \partial P$ such that $\psi(p) = p$ for all $p \in \partial P$, then $\psi^* \theta$ is closed.

(c) Prove Theorem 1.
Hint: use integration to get a contradiction.

We conclude with the following result.

Theorem 2. *Let $D_n := \{x \in \mathbb{R}^n \mid \|x\| \leq 1\}$ denote the closed unit disk in $\mathbb{R}^n$. Any smooth map $\phi : D_n \to D_n$ has a fixed point, that is, $\exists\, p \in D_n$ such that $\phi(p) = p$.*

(d) Prove Theorem 2.
Hint: by contradiction, consider $\psi(p) = \frac{p - \phi(p)}{\|p - \phi(p)\|} \ldots$